A THEOREM IN FACTORIALS.

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Let the indicated product

$$a(a-d)(a-2d)\ \dots\ [a-(n-1)d]$$

be expanded by multiplication into the series

$$a^n+\varphi_1(n)a^{n-1}d+\varphi_2(n)a^{n-2}d^2+\ \dots\ +\varphi_{n-2}(n)a^2d^{n-2}+\varphi_{n-1}(n)ad^{n-1},$$

$\varphi_1(n)$, $\varphi_2(n)$, . . . being functions of n whose form does not depend on the value of n.

Let the symbol ${}_da_n$, whatever the values of a, d, and n, be defined by

$$ {}_da_n \equiv a^n+\varphi_1(n)a^{n-1}d+\varphi_2(n)a^{n-2}d^2+\ \dots . \qquad (A)$$

Then let us consider the following

THEOREM.—${}_d(x+y)_n = {}_dx_n + n\,.\,{}_dx_{n-1}\,.\,{}_dy_1 + \dfrac{n(n-1)}{2!}\,.\,{}_dx_{n-2}\,.\,{}_dy_2$

$$+\frac{n(n-1)(n-2)}{3!}\,.\,{}_dx_{n-3}\,.\,{}_dy_3+\ \dots .$$

I. *Let n be a positive integer.*

In this case, from (A),

$$ {}_d(x+y)_n=(x+y)(x+y-d)(x+y-2d)\ \dots\ [x+y-(n-1)d],$$

$$ {}_dx_n = x(x-d)(x-2d)\ \dots\ [x-(n-1)d],$$

$$ {}_dx_{n-1} = x(x-d)(x-2d)\ \dots\ [x-(n-2)d],$$

. . . .

Let us assume the theorem true for any particular value of n, and multiply both sides by $x+y-nd$. The left side obviously becomes ${}_d(x+y)_{n+1}$. In multiplying the $(r+1)$th term of the right side, separate $x+y-nd$ into $x-(n-r)d$ and $y-rd$. The $(r+1)$th term,

$$\frac{n(n-1)(n-2)\ \dots\ (n-r+1)}{r!}\,.\,{}_dx_{n-r}\,.\,{}_dy_r,$$

multiplied by $x-(n-r)d$ equals

$$\frac{n(n-1)(n-2)\ \dots\ (n-r+1)}{r!}\,.\,{}_dx_{n-r+1}\,.\,{}_dy_r,$$

and multiplied by $y - rd$ equals

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\cdot {}_dx_{n-r}\cdot {}_dy_{r+1}.$$

The rth term treated in like manner will yield the two products

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{(r-1)!}\cdot {}_dx_{n-r+2}\cdot {}_dy_{r-1}$$

and

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{(r-1)!}\cdot {}_dx_{n-r+1}\cdot {}_dy_{r}\cdot$$

The first of the first pair of products and the second of the second pair are the only terms of the product of the right side by $x+y-nd$ in which ${}_dx_{n-r+1}\cdot{}_dy_r$ will appear. The sum of these will therefore be one term of the complete product; viz. the $(r+1)$th term. But this is identical with the $(r+1)$th term in the expansion of ${}_d(x+y)_{n+1}$ by the theorem. Hence if the theorem is true for any particular value of n, it is true for the value greater by unity; but it is evidently true when $n=1$; it is therefore true for any positive integral value of n.

As a particular case, when $d=1$, we have Vandermonde's Theorem.

Again, when $d=0$, we have the Binomial Theorem for the case of a positive integral exponent, since, from (A), ${}_0a_n = a^n$.

II.—*Let n be a negative integer.*

Let $n = -m$. We will first show that

$${}_da_{-m} = \frac{1}{{}_d(a+md)_m}.$$

Now $$ {}_da_r = {}_da_{r-1}\cdot[a-(r-1)d]; \qquad (1)$$

for ${}_da_{r-1}$ expanded by (A) and multiplied by $a-(r-1)d$ becomes the expansion of ${}_da_r$ by (A). Hence,

$${}_da_r = {}_da_{r-s}\cdot{}_d[a-(r-s)d]_s, \qquad (2)$$

s being a positive integer. Now from (A) ${}_da_1 = a$. From (1) ${}_da_1 = {}_da_0\cdot a$. Hence ${}_da_0 = 1$. From (2) ${}_da_0 = {}_da_{-s}\cdot(a+sd)_s$. Hence

$${}_da_{-m} = {}_d(a+md)_m^{-1} \qquad (3)$$

Let $E_d(x+y)_n$ denote the expansion of ${}_d(x+y)_n$ by the theorem, and let $E_d(x+y)_{-m}$ be multiplied by $E_d(x+y+md)_m$. If $E_d(x+y)_{-m}$ is convergent and remains convergent when its negative terms (if there are any) are made positive, the series resulting from multiplication will be convergent, and will equal ${}_d(x+y+md)_m$ multiplied by the value of $E_d(x+y)_{-m}$.*

* See Cauchy's *Analyse Algébrique*, or Chas. Smith's Treatise on Algebra.

But $$E_d(x+y)_{-m} \times E_d(x+y+md)_m = 1,$$
and $$_d(x+y)_{-m} \times {}_d(x+y+md)_m = 1;$$
hence $$_d(x+y)_{-m} = E_d(x+y)_{-m}$$
under the condition above stated.

It may be shown that $E_d(x+y)_{-m}$ is convergent when y is numerically less than $x+d$.

It follows from the above that $E_d(x+y)_{-m}$ is never finite when $_d(x+y)_{-m}$ is infinite.

We can thence prove the series
$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\tfrac{1}{n}+\ldots$$
divergent; for
$$E_1(0-1)_{-1} = {}_10_{-1} + (-1)\,.\,{}_10_{-2}\cdot{}_1(-1)_1 + \frac{(-1)(-2)}{2!}\cdot{}_10_{-3}\cdot{}_1(-1)_2 + \ldots$$
$$= \frac{1}{1} + \frac{1}{1.2} + \frac{1.2}{1.2.3} + \ldots$$
$$= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots,$$
and
$$_1(0-1)_{-1} = {}_10_1^{-1} = 0^{-1} = \infty.$$

Again, the familiar series
$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots$$
is $$E_1(1-1)_{-1}, \text{ and } {}_1(1-1)_{-1} = {}_11_1^{-1} = 1.$$

The theorem is thus seen to be of use in the summation of series.

When $d=0$ we have here the Binomial Theorem for the case of a negative integral exponent.

III. *Let n be any number.*

1. Let $d=0$. The theorem becomes the Binomial Theorem for any index, which has been proved true when the series is convergent; and the series has been proved convergent when y is numerically less than x. We will therefore assume these results.

2. Let d be any number. Let $D_d(x+y)_n$ denote the expansion of $_d(x+y)_n$ by (A). If y be numerically less than x, we can substitute for $(x+y)^n$, $(x+y)^{n-1}$, . . . in $D_d(x+y)_n$ their respective expansions by the Binomial Theorem. Then the coefficient of any power of x in $D_d(x+y)_n$ equals the coefficent of the same power of x in $E_d(x+y)_n$. Hence the theorem is true for any value of n, provided y is numerically less than x.